Differential Geometry I: Curves and surfaces

A. Standard exercises: We will study the oriented curvature of planar curves and its geometric meaning; in what follows, if γ is a regular plane curve, we will denote κ_γ^{or} simply as κ_γ or κ. We will also begin reviewing differential calculus in many variables.

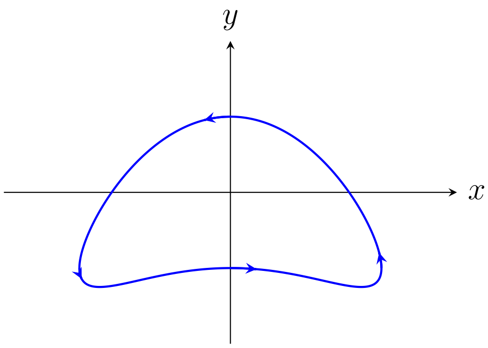
6.1 **(a)** Let γ be a plane curve whose curvature κ is a strictly monotonic function of the arc-length parameter. Can this curve be a closed regular C^2 curve?

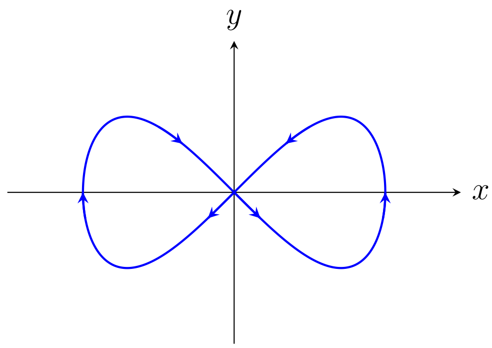
(b) Consider the following plane curves: a circle, an ellipse, and a parabola, each with its natural parametrization. For each of these curves, sketch qualitatively the graph of the function $s \mapsto \kappa(s)$ (this graph is called the *curvature diagram* of the curve).

6.2 Compute the integral

$$\int_\gamma \kappa \, ds$$

for the following closed curves (parametrizaed in the direction indicated):





6.3 Let $\gamma : I \to \mathbb{R}^2$ be a C^3 biregular planar curve. Assume that, for some $t_0 \in I$, $\gamma(t_0)$ is a vertex, i.e. $\frac{d\kappa}{dt}(t_0) = 0$ (this is property of the point $\gamma(t_0)$ is of course independent of parametrization). Show that the *evolute* β of γ is not regular at $t = t_0$. In the case when $\frac{d\kappa}{dt}(t)$ is strictly monotonic in a neighborhood of $t = t_0$, show that the unit tangent to β satisfies

$$\lim_{t \to t_0+} T_\beta(t) = -\lim_{t \to t_0-} T_\beta(t)$$

(in fact, β has a cusp point at $t = t_0$).

6.4 A bit of differential calculus:

(a) Compute the differential (in the Fréchet sense) $d\varphi_A(H)$ of the map $\varphi : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ (recall that $\mathcal{M}_n(\mathbb{R})$ is the space of $n \times n$ real matrices) defined by $\varphi(A) = A^3$, for arbitrary $A, H \in \mathcal{M}_n(\mathbb{R})$. What can be said in the special case where A and H commute?

(b) Let $\phi, \psi : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be two differentiable maps. Prove the following version of the Leibniz rule:

$$d(\phi \cdot \psi)_A(H) = d\phi_A(H)\psi(A) + \phi(A)d\psi_A(H),$$

where $(\phi \cdot \psi)(A) = \phi(A) \cdot \psi(A)$ (matrix product).

(c) Using the previous result, show that if $\phi : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is defined by $\phi(A) = A^{-1}$, then

$$d\phi_A(H) = -A^{-1}HA^{-1}.$$

6.5 Prove that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(y_1, y_2) = f(x_1, x_2) = (x_1 \cos(x_2),\, x_2 - x_1x_2)$$

is a diffeomorphism in a neighborhood of $(0, 0)$.

6.6 **(a)** Recall the definition of a curvilinear coordinate system.

(b) Prove the following statement or find a counterexample: If $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are two curvilinear coordinate systems on an open set $U \subset \mathbb{R}^2$ and if $y_2 = x_2$, then

$$\frac{\partial}{\partial y_2} = \frac{\partial}{\partial x_2}.$$

6.7 Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be two distinct points in $\mathbb{R}^2$. Prove that the functions

$$u(x, y) = d\big((x, y), (p_1, p_2)\big), \quad v(x, y) = d\big((x, y), (q_1, q_2)\big)$$

(where $d(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R}^2$) define a C^∞ curvilinear coordinate system in each of the two half-planes bounded by the line passing through p and q. Describe the coordinate lines.

B. Bonus exercises:

6.8 **(a)** Recall the conditions under which one can define the osculating circle of a curve $\alpha : I \to \mathbb{R}^n$ at a given point. Recall the definition of the osculating circle.

(b) How can one find the center and radius of the osculating circle at a given point of the curve? Specify the plane in which this circle lies.

(c) Prove the following result: Let $\alpha : I \to \mathbb{R}^2$ be a C^3 plane curve whose curvature is positive and strictly increasing. Then the osculating circles $\mathcal{C}(s)$ of α are nested as follows: if $s_1 < s_2$, then $\mathcal{C}(s_2)$ is contained in the disk bounded by $\mathcal{C}(s_1)$.

Hint: First show that the radius $\rho(s)$ of $\mathcal{C}(s)$ is a decreasing function of s. Then show that the distance between the centers of $\mathcal{C}(s_1)$ and $\mathcal{C}(s_2)$ is less than the difference of their radii (why does this answer the question?). To justify this last statement, it is useful to assume that α is parametrized by arc length and to compute the velocity of $s \mapsto c(s)$ (the derivative of the center of $\mathcal{C}(s)$ can be easily computed in the Frenet frame).

6.9 **(a)** Let $\gamma : [0, \infty) \to \mathbb{R}^2$ be a C^3 plane curve of infinite length whose curvature is positive and strictly increasing. Prove that the trace of this curve is bounded. Can you give an explicit bound (i.e., a constant C depending maybe on the minimum curvature such that $\|\gamma(s) - \gamma(0)\| \leqslant C$ for all s)?

(b) Show by example that the monotonicity assumption on the curvature is necessary. More precisely, show that there exists a curve whose curvature satisfies $\kappa(s) \geqslant a > 0$ for all s but which is not bounded. (It is not necessary to produce an explicit formula; a drawing suffices.)

Hint for (a): Think about Exercise 6.8(c).

6.10 Let $\gamma(s) = (x(s), y(s)) \in \mathbb{R}^2$ denote the clothoid curve parametrized by arc length (see Exercise 4.6). Do you think that the limit

$$\lim_{s \to \infty} \gamma(s) \in \mathbb{R}^2$$

exists?

(*You are asked for a geometric argument, not to compute or analyze the Fresnel integrals. Exercise 6.8 (c) is useful here.*)